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On a Mode Superposition Method for Fluid-Structure Interaction Problems

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Abstract

In this paper, we present a mode superposition method for fluid-structure interaction problems based on velocity potential and pressure formulation for acoustic fluids and displacement formulation for solids. The effectiveness of the proposed numerical procedure is demonstrated in a test example with seismic ground motion effects.

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1 Introduction

Fluid-structure interaction systems are widely used in various industrial applications [1] [2]. A number of finite element formulations have been proposed to model an acoustic fluid for the analysis of fluid-structure interaction problems, namely, the displacement formulation (see Bathe and Hahn [3], Hamdi, et al. [4], Olson and Bathe [5], Bathe et al. [6], Wang and Bathe [7] [8]), the displacement potential and pressure formulation, and the velocity potential and pressure formulation (Morand and Ohayon [9], Everstine [10], Olson and Bathe [11], Felippa and Ohayon [12], MacNeal et al. [13]). A recent review of various approaches for fluid-structure interaction problems is available in Ref. [14].

The eigenvalue solution techniques for the $P-\phi-U$ finite element formulation (velocity potential and pressure formulation for acoustic fluids and displacement formulation for solids) have been developed by Olson et al. [11] [15]. In this paper, we propose a mode superposition method as an extension of the previous research of Refs. [11] [15]. A simple numerical model is used to verify the implementation and to demonstrate the capability of the proposed procedure.

In the following section, we briefly summarize the governing equations. We discuss in Section 3 the proposed mode superposition method. In Section 4, we present a treatment of the ground motion effects. Some numerical validations will be considered in Section 5.

2 Velocity Potential-based Formulations

We assume an inviscid, irrotational compressible fluid with small motions and no gravity effects. In the $\phi-U$ formulation, we use the velocity potential as the state variable for fluids and the displacement for solids. In the $P-\phi-U$ formulation, we

replace one velocity potential unknown with a pressure unknown for each fluid region to eliminate the zero frequency mode. However, both potential-based formulations will give the same natural frequencies of the fluid-structure system except at zero.

For the structure domain V_s with its natural boundary S_s and the fluid-structure interface S_{fs}, the variational indicator is defined as

$$\begin{aligned}(V.I.)_s &= \int_{t_1}^{t_2}\{\frac{1}{2}\int_{V_s}\boldsymbol{\epsilon}^T\mathbf{C}_s\boldsymbol{\epsilon}dV - \frac{1}{2}\int_{V_s}\rho_s\dot{\mathbf{u}}^T\dot{\mathbf{u}}dV \\ &\quad - \int_{S_{fs}}(\mathbf{u}^{S_{fs}})^T\boldsymbol{f}^{S_{fs}}dS - \int_{S_s}(\mathbf{u}^{S_s})^T\boldsymbol{f}^{S_s}dS - \int_{V_s}\mathbf{u}^T\mathbf{b}_s dV\}dt \qquad (1)\end{aligned}$$

where $\mathbf{C}_s$, $\boldsymbol{\epsilon}$, ρ_s, $\mathbf{u}$, $\boldsymbol{f}$, and $\mathbf{b}_s$ stand for the stress-strain material matrix, strain tensor, density of solid, displacement vector, surface (or interface) force vector, and body force vector, respectively.

For the fluid domain V_f with the fluid-structure interface S_{fs}, the variational indicator has the form

$$\begin{aligned}(V.I.)_f &= \int_{t_1}^{t_2}\{\frac{1}{2}\int_{V_f}\frac{1}{\beta}(P-\rho_f\dot{\phi})^2dV - \frac{1}{2}\int_{V_f}\rho_f(\nabla\phi)^2dV \\ &\quad - \int_{S_{fs}}(P-\rho_f\dot{\phi}^{S_{fs}})\mathbf{u}^{S_{fs}}\cdot\mathbf{n}dS\}dt \qquad (2)\end{aligned}$$

where ϕ is the velocity potential, P is the pressure, β is the bulk modulus, ρ_f is the fluid density, $\mathbf{n}$ is the unit outward normal vector from solids, and consequently $\boldsymbol{f}^{S_{fs}} = -(P-\rho_f\dot{\phi}^{S_{fs}})\mathbf{n}$. Note that only one hydrostatic pressure unknown in the $P-\phi-U$ formulation is used to replace one nodal velocity potential in the $\phi-U$ formulation for each fluid region. If we have k separate fluid domains, we need to use k independent hydrostatic pressure unknowns.

Since all variations vanish at $t=t_1$ and $t=t_2$, we obtain

$$\int_{V_s} \rho_s \delta \mathbf{u}^T \ddot{\mathbf{u}} dV + \int_{V_s} \delta \boldsymbol{\epsilon}^T \mathbf{C}_s \boldsymbol{\epsilon} dV - \int_{S_{fs}} (\delta \mathbf{u}^{S_{fs}})^T \boldsymbol{f}^{S_{fs}} dS$$
$$- \int_{S_s} (\delta \mathbf{u}^{S_s})^T \boldsymbol{f}^{S_s} dS - \int_{V_s} \delta \mathbf{u}^T \mathbf{b}_s dV = 0 \tag{3}$$

$$\int_{V_f} \frac{P}{\beta} \delta P dV - \int_{V_f} \frac{\rho_f}{\beta} \dot{\phi} \delta P dV + \int_{V_f} \frac{\dot{P}}{\beta} \rho_f \delta \phi dV - \int_{V_f} \rho_f (\nabla \delta \phi) \cdot (\nabla \phi) dV$$
$$- \int_{V_f} \frac{{\rho_f}^2}{\beta} \ddot{\phi} \delta \phi dV - \int_{S_{fs}} \mathbf{u}^{S_{fs}} \cdot \mathbf{n} \delta P dS - \int_{S_{fs}} \rho_f \dot{\mathbf{u}}^{S_{fs}} \cdot \mathbf{n} \delta \phi^{S_{fs}} dS = 0. \tag{4}$$

Using the standard finite element procedure, and considering a typical element, we have

$$\begin{array}{rrcl@{\qquad}rcl} \text{for solids:} & \mathbf{u} & = & \mathbf{H}\,\mathbf{U}, & \boldsymbol{\epsilon} & = & \mathbf{B}\mathbf{U}; \\ \text{for fluids:} & \phi & = & \mathbf{h}\,\boldsymbol{\Phi}, & \nabla\phi & = & \mathbf{D}\,\boldsymbol{\Phi}; \\ \text{for f-s interfaces:} & \mathbf{u}^{S_{fs}} \cdot \mathbf{n} & = & \mathbf{b}\,\mathbf{U}, & \phi^{S_{fs}} & = & \mathbf{a}\,\boldsymbol{\Phi}. \end{array} \tag{5}$$

The matrix equations derived from Eqs. (3) and (4) are

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{M}_{ff} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}} \\ \ddot{\mathbf{P}} \\ \ddot{\boldsymbol{\Phi}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{C}_{fs}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{pf} \\ \mathbf{C}_{fs} & \mathbf{C}_{pf}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{U}} \\ \dot{\mathbf{P}} \\ \dot{\boldsymbol{\Phi}} \end{Bmatrix}$$
$$+ \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{ps}^T & \mathbf{0} \\ \mathbf{K}_{ps} & \mathbf{K}_{pp} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{K}_{ff} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Phi} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_s \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{6}$$

where

$$\begin{array}{rcl@{\qquad}rcl} \mathbf{K}_{ss} & = & \int \mathbf{B}^T \mathbf{C}_s \mathbf{B} dV, & \mathbf{M}_{ss} & = & \int \rho_s \mathbf{H}^T \mathbf{H} dV; \\ \mathbf{C}_{pf} & = & \int \frac{\rho_f}{\beta} \mathbf{h} dV, & \mathbf{M}_{ff} & = & \int \frac{\rho_f^2}{\beta} \mathbf{h}^T \mathbf{h} dV; \\ \mathbf{K}_{pp} & = & -\int \frac{1}{\beta} dV, & \mathbf{C}_{fs}^T & = & -\int \rho_f \mathbf{b}^T \mathbf{a} dS; \\ \mathbf{K}_{ff} & = & \int \rho_f \mathbf{D}^T \mathbf{D} dV, & \mathbf{K}_{ps}^T & = & \int \mathbf{b}^T dS; \\ \mathbf{R}_s & = & \int \mathbf{H}^T \mathbf{b}_s dV + \int \mathbf{H}^{S_s T} \boldsymbol{f}^{S_s} dS. & & & \end{array} \tag{7}$$

For the fluid element in which one velocity potential unknown is replaced by the pressure unknown (without loss of generality, we assume the Nth nodal velocity potential unknown of a typical N-node fluid element is replaced by a pressure unknown), the discretization relations become

$$\phi = \mathbf{h}' \boldsymbol{\Phi}, \qquad \nabla\phi = \mathbf{D}' \boldsymbol{\Phi}, \qquad \phi^{S_{fs}} = \mathbf{a}' \boldsymbol{\Phi}; \tag{8}$$

and consequently, the corresponding matrices for the element are given as:

$$\begin{aligned}
\mathbf{C}_{pf} &= \int \frac{\rho_f}{\beta} \mathbf{h}' dV, & \mathbf{M}_{ff} &= \int \frac{\rho_f^2}{\beta} \mathbf{h}'^T \mathbf{h}' dV; \\
\mathbf{K}_{pp} &= -\int \frac{1}{\beta} dV, & \mathbf{C}_{fs}^T &= -\int \rho_f \mathbf{b}^T \mathbf{a}' dS; \\
\mathbf{K}_{ff} &= \int \rho_f \mathbf{D}'^T \mathbf{D}' dV, & \mathbf{K}_{ps}^T &= \int \mathbf{b}^T dS.
\end{aligned} \tag{9}$$

From the kinematic boundary conditions along the fluid-structure interface, we recognize that physically $\boldsymbol{\Phi}$ has a $\pi/2$ phase shift from $\mathbf{U}$ ($\dot{\mathbf{u}} \sim \boldsymbol{\nabla}\phi$). Therefore, the corresponding eigensolution (sinusoidal in time) takes the form $\mathbf{X} = \mathbf{X}_m e^{\lambda_m t}$, where $\mathbf{X}_m^T = (\mathbf{U}_m^T \ \mathbf{P}_m^T \ i\boldsymbol{\Phi}_m^T)$, and $\mathbf{U}_m$, $\mathbf{P}_m$ and $\boldsymbol{\Phi}_m$ are real values. Furthermore, with the sign change of the second equation in Eq. (6), we obtain

$$a_m \lambda_m^2 + b_m \lambda_m + c_m = 0 \tag{10}$$

where

$$\begin{aligned}
a_m &= \mathbf{U}_m^T \mathbf{M}_{ss} \mathbf{U}_m + \boldsymbol{\Phi}_m^T \mathbf{M}_{ff} \boldsymbol{\Phi}_m \\
b_m &= 2i \mathbf{U}_m^T \mathbf{C}_{fs}^T \boldsymbol{\Phi}_m \\
c_m &= \mathbf{U}_m^T \mathbf{K}_{ss} \mathbf{U}_m + \boldsymbol{\Phi}_m^T \mathbf{K}_{ff} \boldsymbol{\Phi}_m + \mathbf{P}_m^T (-\mathbf{K}_{pp}) \mathbf{P}_m.
\end{aligned}$$

Since $\mathbf{M}_{ss}$, $\mathbf{M}_{ff}$, $-\mathbf{K}_{pp}$, $\mathbf{K}_{ff}$, and $\mathbf{K}_{ss}$ are positive definite matrices, a_m and c_m should be positive and Eq. (10) gives pure imaginary eigenvalues of the coupled fluid-structure system.

By assigning $\lambda_m = i\omega_m$, $m = 1, 2, ..., n$, where n is the total number of degrees of freedom, and substituting eigensolution $\mathbf{X} = \mathbf{X}_m e^{i\omega_m t}$ into the left side of Eq. (6), we get

$$-\omega_m^2 \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{ff} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_m \\ \mathbf{P}_m \\ \boldsymbol{\Phi}_m \end{Bmatrix} - \omega_m \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{C}_{fs}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{pf} \\ \mathbf{C}_{fs} & \mathbf{C}_{pf}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_m \\ \mathbf{P}_m \\ \boldsymbol{\Phi}_m \end{Bmatrix}$$

$$+ \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{ps}^T & \mathbf{0} \\ \mathbf{K}_{ps} & \mathbf{K}_{pp} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{ff} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_m \\ \mathbf{P}_m \\ \boldsymbol{\Phi}_m \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{11}$$

In the $\phi - U$ formulation, we obtain an equation similar to Eq. (6),

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ff} & -\mathbf{M}_{af}^T \\ \mathbf{0} & -\mathbf{M}_{af} & -\mathbf{M}_{aa} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}} \\ \ddot{\boldsymbol{\Phi}} \\ \ddot{\boldsymbol{\Phi}}_a \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{C}_{fs}^T & \mathbf{C}_{as}^T \\ \mathbf{C}_{fs} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{as} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{U}} \\ \dot{\boldsymbol{\Phi}} \\ \dot{\boldsymbol{\Phi}}_a \end{Bmatrix}$$

$$+ \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_{ff} & -\mathbf{K}_{af}^T \\ \mathbf{0} & -\mathbf{K}_{af} & -\mathbf{K}_{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \boldsymbol{\Phi} \\ \boldsymbol{\Phi}_a \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_s \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{12}$$

where $\boldsymbol{\Phi}_a$ denotes the velocity potentials that will be replaced by pressure unknowns. Similarly, the mth eigensolution of Eq. (12) can be written as $\mathbf{X} = \mathbf{X}_m e^{\lambda_m t}$, where $\mathbf{X}_m^T = (\mathbf{U}_m^T \; i\boldsymbol{\Phi}_m^{'T} \; i(\boldsymbol{\Phi}_a)_m^T)$. Assign $\lambda_m = i\omega_m$, we obtain

$$-\omega_m^2 \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{ff} & \mathbf{M}_{af}^T \\ \mathbf{0} & \mathbf{M}_{af} & \mathbf{M}_{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_m \\ \boldsymbol{\Phi}_m^{'} \\ (\boldsymbol{\Phi}_a)_m \end{Bmatrix} - \omega_m \begin{bmatrix} \mathbf{0} & \mathbf{C}_{fs}^T & \mathbf{C}_{as}^T \\ \mathbf{C}_{fs} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{as} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_m \\ \boldsymbol{\Phi}_m^{'} \\ (\boldsymbol{\Phi}_a)_m \end{Bmatrix}$$

$$+ \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{ff} & \mathbf{K}_{af}^T \\ \mathbf{0} & \mathbf{K}_{af} & \mathbf{K}_{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_m \\ \boldsymbol{\Phi}_m^{'} \\ (\boldsymbol{\Phi}_a)_m \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{13}$$

and it can be shown that Eq. (13) has the same eigenvalues as Eq. (6) except at zero (refer to Appendix A and Refs. [11] [15]).

3 Mode Superposition Method

Determinant search and subspace iteration methods can be used to find the needed real eigenvalues ω_m from Eq. (11), where $m = 1, ..., p$ and p stands for the number of modes below the cut-off frequency [11] [15]. Note that the number of negative elements in the matrix $\mathbf{D}$ of the $\mathbf{L}\ \mathbf{D}\ \mathbf{L}^T$ factorization of $\mathbf{K} - \omega_m \mathbf{C} - \omega_m^2 \mathbf{M}$ is equal to the number of eigenvalues below ω_m^2 plus the number of pressure unknowns.

With $\mathbf{X}^T = (\mathbf{U}^T\ \mathbf{P}^T\ \boldsymbol{\Phi}^T)$ and $\mathbf{F}^T = (\mathbf{R}_s^T\ \mathbf{0}^T\ \mathbf{0}^T)$, Eq. (6) can be rewritten as follows:

$$\mathbf{A}\ \mathbf{Y} + \mathbf{B}\ \dot{\mathbf{Y}} = \boldsymbol{\mathcal{F}} \tag{14}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} \mathbf{X} \\ \dot{\mathbf{X}} \end{pmatrix};$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\mathcal{F}} = \begin{pmatrix} \mathbf{F} \\ \mathbf{0} \end{pmatrix}.$$

Without loss of generality, if we assume the system has distinct eigenvalues, the following orthogonal relationships hold:

$$\mathbf{Y}_m^T \mathbf{B}\ \mathbf{Y}_k = \begin{cases} 0 & \text{if } m \neq k \\ b_m + 2a_m\lambda_m & \text{if } m = k \end{cases} \tag{15}$$

$$\mathbf{Y}_m^T \mathbf{A}\ \mathbf{Y}_k = \begin{cases} 0 & \text{if } m \neq k \\ c_m - a_m\lambda_m^2 & \text{if } m = k. \end{cases} \tag{16}$$

For multiple eigenvalues, we can always construct independent eigenvector pairs that still have the orthogonality properties [16].

If we write the solution of Eq. (14) as $\mathbf{Y}(t) = \mathbf{Y}\mathbf{Q}(t) = \sum_{m=1}^{2n} \mathbf{Y}_m q_m(t)$, where $\mathbf{Y} = (\mathbf{Y}_1\ \mathbf{Y}_2\ ...\ \mathbf{Y}_{2n})$ is the mode shape matrix, and $\mathbf{Q}^T(t) = (q_1\ q_2\ ...\ q_{2n})$ is

the generalized coordinate vector, Eq. (14) can be decoupled by employing the orthogonality relations,

$$\dot{q}_m + p_m q_m = h_m \tag{17}$$

where $m = 1, 2, ..., 2n$ and

$$\begin{array}{rcl} p_m & = & (c_m - a_m\lambda_m^2)/(b_m + 2a_m\lambda_m) = -\lambda_m, \\ h_m & = & \mathbf{Y}_m^T \, \boldsymbol{\mathcal{F}}/(b_m + 2a_m\lambda_m). \end{array} \tag{18}$$

Note that the initial vector $\mathbf{Y}_0^T = (\mathbf{U}_0^T \; \mathbf{P}_0^T \; \boldsymbol{\Phi}_0^T \; \dot{\mathbf{U}}_0^T \; \dot{\mathbf{P}}_0^T \; \dot{\boldsymbol{\Phi}}_0^T)$ has to satisfy the second equation of Eq. (6), and for Eq. (17), the initial condition is written as $q_m(0) = \mathbf{Y}_m^T \mathbf{B} \, \mathbf{Y}_0/(b_m + 2a_m\lambda_m)$. Reconstructing the complex eigenvector and eigenvalue conjugate pairs yields

$$\begin{array}{rcllllll} \mathbf{Y}_{2m-1}^T & = & (\mathbf{U}_m^T & \mathbf{P}_m^T & i\boldsymbol{\Phi}_m^T & i\omega_m\mathbf{U}_m^T & i\omega_m\mathbf{P}_m^T & -\omega_m\boldsymbol{\Phi}_m^T) \\ \mathbf{Y}_{2m}^T & = & (\mathbf{U}_m^T & \mathbf{P}_m^T & -i\boldsymbol{\Phi}_m^T & -i\omega_m\mathbf{U}_m^T & -i\omega_m\mathbf{P}_m^T & -\omega_m\boldsymbol{\Phi}_m^T) \end{array} \tag{19}$$

with $m = 1, 2, ..., n$ and moreover,

$$\begin{array}{llll} q_{(2m-1)}(0) & = \mathbf{Y}_{(2m-1)}^T \mathbf{V}_o/(b_m + 2\lambda_m a_m) & = (\beta_m - \alpha_m i)/(\gamma_m + \xi_m) & = A_m + B_m i \\ q_{(2m)}(0) & = -\mathbf{Y}_{2m}^T \mathbf{V}_o/(b_m + 2\lambda_m a_m) & = (\beta_m + \alpha_m i)/(\gamma_m + \xi_m) & = A_m - B_m i \end{array}$$

where $A_m = \beta_m/(\gamma_m + \xi_m)$, $B_m = -\alpha_m/(\gamma_m + \xi_m)$, and

$$\mathbf{V}_o \;=\; \mathbf{B} \; \mathbf{Y}_o \;=\; \left\{ \begin{array}{c} \mathbf{V}_o^{(u)} \\ \mathbf{V}_o^{(p)} \\ \mathbf{V}_o^{(\phi)} \\ \mathbf{V}_o^{(\dot{u})} \\ \mathbf{V}_o^{(\dot{p})} \\ \mathbf{V}_o^{(\dot{\phi})} \end{array} \right\}$$

$$\alpha_m = (\mathbf{U}_m^T \mathbf{V}_o^{(u)} + \mathbf{P}_m^T \mathbf{V}_o^{(p)} - \omega_m \boldsymbol{\Phi}_m^T \mathbf{V}_o^{(\dot{\Phi})}), \qquad \gamma_m = 2(\boldsymbol{\Phi}_m^T \mathbf{C}_{fs} \mathbf{U}_m + \boldsymbol{\Phi}_m^T \mathbf{C}_{pf}^T \mathbf{P}_m);$$

$$\beta_m = (\boldsymbol{\Phi}_m^T \mathbf{V}_o^{(\phi)} + \omega_m \mathbf{U}_m^T \mathbf{V}_o^{(\dot{u})} + \omega_m \mathbf{P}_m^T \mathbf{V}_o^{(\dot{p})}), \quad \xi_m = 2\omega_m(\mathbf{U}_m^T \mathbf{M}_{ss} \mathbf{U}_m + \boldsymbol{\Phi}_m^T \mathbf{M}_{ff} \boldsymbol{\Phi}_m).$$

In addition, with

$$\boldsymbol{\mathcal{F}} = \left\{ \begin{array}{c} \mathbf{F}_o^{(u)} \\ \mathbf{F}_o^{(p)} \\ \mathbf{F}_o^{(\phi)} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array} \right\},$$

we derive from Eq. (18),

$$h_{(2m-1)} = C_m + D_m i, \qquad h_{2m} = C_m - D_m i$$

where

$$D_m = -(\mathbf{U}_m^T \mathbf{F}^{(u)} + \mathbf{P}_m^T \mathbf{F}^{(p)})/(\gamma_m + \xi_m), \qquad C_m = \boldsymbol{\Phi}_m^T \mathbf{F}^{(\phi)}/(\gamma_m + \xi_m).$$

Finally, the solution of Eq. (17) can be written as:

$$q_m(t) = \int_0^t e^{p_m(\tau - t)} h_m(\tau) d\tau + q_m(0) e^{-p_m t} \tag{20}$$

and the solution of Eq. (14) is in the form

$$\left\{ \begin{array}{c} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Phi} \\ \dot{\mathbf{U}} \\ \dot{\mathbf{P}} \\ \dot{\boldsymbol{\Phi}} \end{array} \right\} = \sum_{m=1}^{n} \left\{ \begin{array}{cc} \mathbf{U}_m & \{F_m^u(t) + 2A_m \cos\omega_m t - 2B_m \sin\omega_m t\} \\ \mathbf{P}_m & \{F_m^p(t) + 2A_m \cos\omega_m t - 2B_m \sin\omega_m t\} \\ \boldsymbol{\Phi}_m & \{F_m^\phi(t) - 2A_m \sin\omega_m t - 2B_m \cos\omega_m t\} \\ \omega_m \mathbf{U}_m & \{F_m^{\dot{u}}(t) - 2A_m \sin\omega_m t - 2B_m \cos\omega_m t\} \\ \omega_m \mathbf{P}_m & \{F_m^{\dot{p}}(t) - 2A_m \sin\omega_m t - 2B_m \cos\omega_m t\} \\ \omega_m \boldsymbol{\Phi}_m & \{F_m^{\dot{\phi}}(t) - 2A_m \cos\omega_m t + 2B_m \sin\omega_m t\} \end{array} \right\} \tag{21}$$

with

$$\begin{aligned} F_m^{\dot{\phi}}(t) &= -\int_0^t (2C_m(\tau)\cos\omega_m(\tau - t) + 2D_m(\tau)\sin\omega_m(\tau - t))d\tau \\ F_m^u(t) = F_m^p(t) &= \int_0^t (2C_m(\tau)\cos\omega_m(\tau - t) + 2D_m(\tau)\sin\omega_m(\tau - t))d\tau \\ F_m^\phi(t) = F_m^{\dot{u}}(t) = F_m^{\dot{p}}(t) &= \int_0^t (2C_m(\tau)\sin\omega_m(\tau - t) - 2D_m(\tau)\cos\omega_m(\tau - t))d\tau. \end{aligned} \tag{22}$$

4 Ground Motion Effects

It is an important aspect to incorporate ground motion effects in fluid-structure systems. One of the main application areas is the design of liquid storage tanks. To include the relative fluid motion in the system, we modify the variational indicator for fluids as follows:

$$\begin{aligned}(V.I.)_f &= \int_{t_1}^{t_2}\{\frac{1}{2}\int_{V_f}\frac{1}{\beta}(P-\rho_f\dot{\phi})^2dV-\frac{1}{2}\int_{V_f}\rho_f(\nabla\phi+\dot{\mathbf{u}}_g)^2dV \\ &\quad -\int_{S_{fs}}(P-\rho_f\dot{\phi}^{S_{fs}})\mathbf{u}^{S_{fs}}\cdot\mathbf{n}dS\}dt, \end{aligned} \tag{23}$$

where the displacement $\mathbf{u}^{S_{fs}}$ and the potential ϕ represent the relative motions, and the ground velocity $\dot{\mathbf{u}}_g$ is assumed to be a known quantity. Furthermore, we derive similarly the dynamic equations of the fluid-structure system, with $\mathbf{G}=\int\rho_f\mathbf{D}^TdV$,

$$\begin{bmatrix}\mathbf{M}_{ss} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{M}_{ff}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{U}}\\ \ddot{\mathbf{P}}\\ \ddot{\boldsymbol{\Phi}}\end{Bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{C}_{fs}^T\\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{pf}\\ \mathbf{C}_{fs} & \mathbf{C}_{pf}^T & \mathbf{0}\end{bmatrix}\begin{Bmatrix}\dot{\mathbf{U}}\\ \dot{\mathbf{P}}\\ \dot{\boldsymbol{\Phi}}\end{Bmatrix}$$

$$+\begin{bmatrix}\mathbf{K}_{ss} & \mathbf{K}_{ps}^T & \mathbf{0}\\ \mathbf{K}_{ps} & \mathbf{K}_{pp} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{K}_{ff}\end{bmatrix}\begin{Bmatrix}\mathbf{U}\\ \mathbf{P}\\ \boldsymbol{\Phi}\end{Bmatrix}=\begin{Bmatrix}\mathbf{R}_s-\mathbf{M}_{ss}\ddot{\mathbf{U}}_g\\ \mathbf{0}\\ \mathbf{G}\dot{\mathbf{u}}_g\end{Bmatrix}. \tag{24}$$

5 Numerical Examples

For didactic reasons, a simple acoustic fluid-structure interaction model (depicted in Fig. 1) is used to compare the proposed mode superposition method with the direct time integration method (in this paper, we use the Newmark Method). No physical damping is considered. The cross section area A, the spring constant k_o, and the rigid plate mass m_o are assigned to be 1.0 m^2, 1.0×10^7 N/m, and 1000.0 kg,

respectively. The other physical parameters are given as follows: $L = 10.0$ m; $\beta = 2.1 \times 10^9$ Pa; and $\rho = 1000.0$ kg/m^3.

From the solution of the coupled system eigenproblems, we obtain the first three nonzero natural frequencies: $\omega_1 = 211.8$, $\omega_2 = 744.0$, and $\omega_3 = 1677.4$ rad/sec. In the $\phi - U$ formulation, for the coupled system with one closed fluid domain, we have one zero frequency corresponding to $U_o = 0$ and constant ϕ_i over the fluid region (refer to Appendix A). We notice that in this fluid-structure system, due to the fluid mass and compressibility, the lowest nonzero frequency of the coupled fluid-structure system is about 112 percent higher that the natural frequency of the piston/spring system without the fluid coupling.

In test case one, an initial positive displacement of the plate ($U_o = 0.2$ cm), and an initial fluid pressure ($p = -4.2 \times 10^7$ Pa) are applied to the system. In test case two, in addition to initial conditions in case one, an excitation force $R(t) = 1.0 \times 10^8 sin(200\pi t)$ N is applied to the plate. In test case three, a gound motion, $\ddot{u}_g = 20.0^2 \pi cos(100\pi t) + 8.0^2 \pi cos(200\pi t)$ m/sec^2, is applied to the *whole* fluid-structure system.

For the three test cases, the comparisons of the displacement and velocity at position A are shown in Figs. 2, 3, and 4. As can be seen, the more modes we use, the closer the mode superposition solutions are to the direct time integration results. In addition, it is also illustrated in Figs. 3 and 4, that if the excitation forces or ground motions have certain frequencies closer to the coupled natural modes, as in test case two, the excitation force frequency (200.0 Hz) is close to the second mode (211.8 Hz), these modes will have the dominating contribution in the mode superposition method.

6 Conclusions

The $\phi-U$ or $P-\phi-U$ formulations for fluids provide an excellent alternative to the displacement-based fluid finite element formulations [6] [7] [8], and they are easy to implement in displacement-based finite element packages for solids and structures. The mode superposition method outlined in this paper provides the useful tool in the corresponding spectrum analysis of fluid-structure interaction problems. Of course, the advantage of the displacement-based fluid formulation is that the effects of gravity and large fluid motions can be directly included as in the analysis of solids. For the potential formulations for fluids, by employing different formulations for solids and fluids, as expected, special care must be taken for the fluid-structure interfaces [11] [14].

With the proposed mode superposition method, for the generalized loads containing only certain frequencies, if the initial displacements and velocities are the linear combinations of a few modes, we could select a fairly small $p << n$ and drastically reduce the computation efforts compared with using the direct integration method. To improve the solution accuracy, the following static correction can be applied,

$$\mathbf{A}\ \triangle \mathbf{Y} = \triangle \boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}} - \sum_{m=1}^{p} \mathbf{Y}_m^T \boldsymbol{\mathcal{F}} (\mathbf{B}\ \mathbf{Y}_m)/(b_m + 2a_m\lambda_m). \tag{25}$$

In the seismic analysis, the earthquake loading in some cases only consists of the lowest few modes, although the order of the system n may be very large. However, in addition to the ground accelerations, the ground velocities are needed in Eq. (24).

When we extend the proposed method to the study of dissipative systems, it is worthy of mention that the fluid-structure coupling discussed in this paper contributes only to the imaginary part of the natural frequencies. Therefore, the damp-

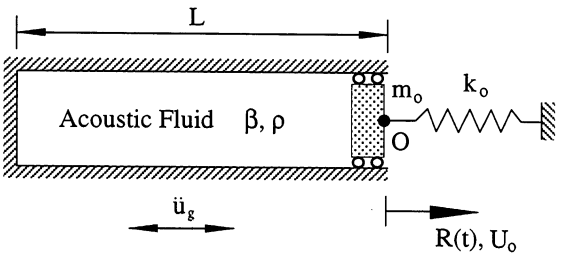

Figure 1: The acoustic fluid in a cavity interacting with a piston/spring system.

ing treatment of the solids will be the same as discussed in Refs. [16] [17].

7 Acknowledgment

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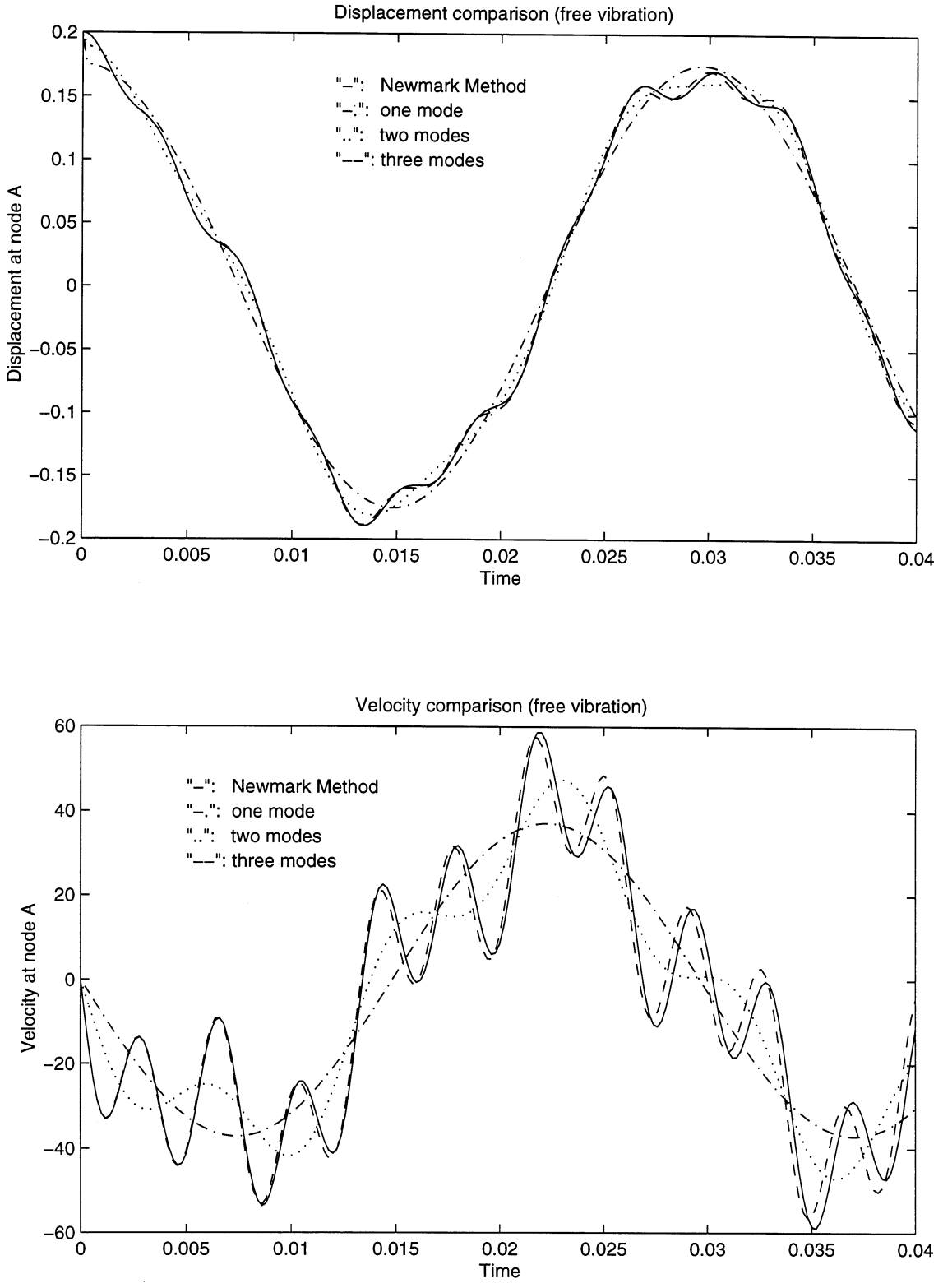

Figure 2: Test case one displacement and velocity comparisons.

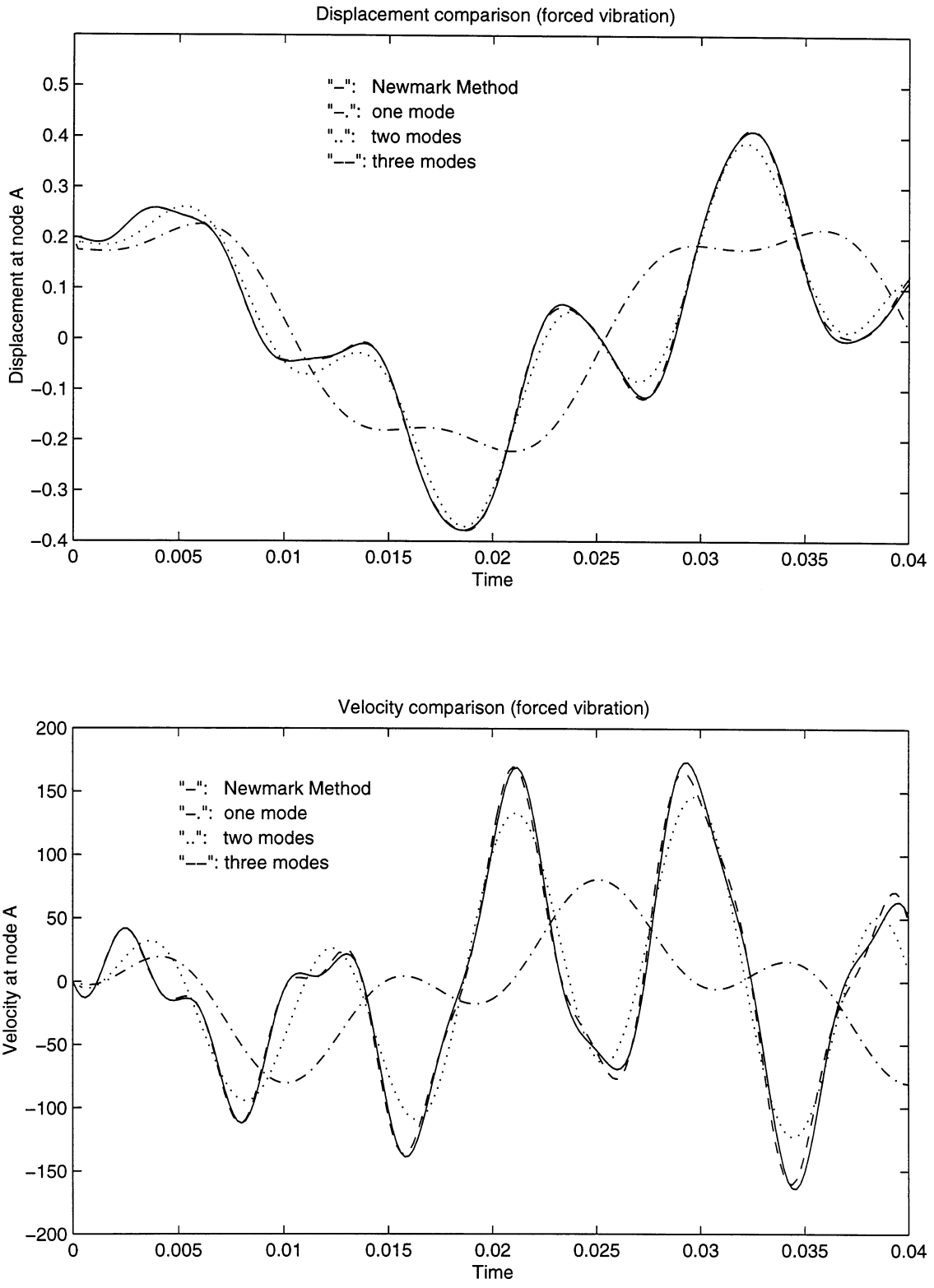

Figure 3: Test case two displacement and velocity comparisons.

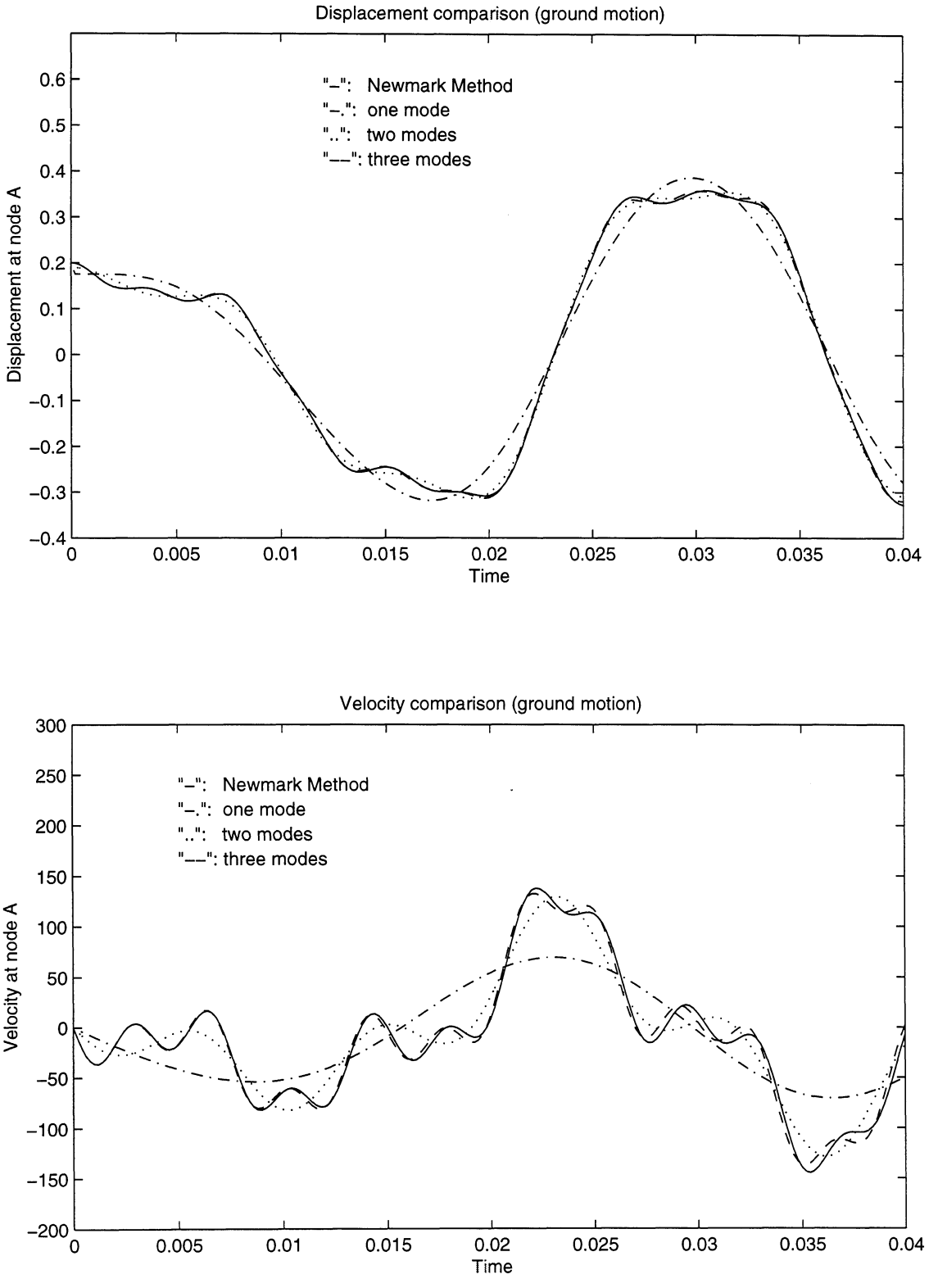

Figure 4: Test case three displacement and velocity comparisons.

Appendix A: $P-\phi-U$ and $\phi-U$ formulations

The $\phi-U$ formulation gives Eq. (13), whereas the $P-\phi-U$ formulation takes the form,

$$-\omega_m^2\begin{bmatrix}\mathbf{M}_{ss} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{ff} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{Bmatrix}\mathbf{U}_m\\ \boldsymbol{\Phi}_m\\ \mathbf{P}_m\end{Bmatrix}-\omega_m\begin{bmatrix}\mathbf{0} & \mathbf{C}_{fs}^T & \mathbf{0}\\ \mathbf{C}_{fs} & \mathbf{0} & \mathbf{B}_p^T\\ \mathbf{0} & \mathbf{B}_p & \mathbf{0}\end{bmatrix}\begin{Bmatrix}\mathbf{U}_m\\ \boldsymbol{\Phi}_m\\ \mathbf{P}_m\end{Bmatrix}$$
$$+\begin{bmatrix}\mathbf{K}_{ss} & \mathbf{0} & \mathbf{A}_p^T\\ \mathbf{0} & \mathbf{K}_{ff} & \mathbf{0}\\ \mathbf{A}_p & \mathbf{0} & \mathbf{D}_p\end{bmatrix}\begin{Bmatrix}\mathbf{U}_m\\ \boldsymbol{\Phi}_m\\ \mathbf{P}_m\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{Bmatrix}. \qquad (A.1)$$

To illustrate that by replacing one nodal velocity potential unknown in the $\phi-U$ formulation with one pressure unknown in the $P-\phi-U$ formulation for each separate fluid domain, we get the same eigenvalues (except at zero) in Eqs. (13) and (A.1), for simplicity, we consider only one fluid domain, i.e., $\mathbf{P}_m=[P]$. The analogy to the case involving several fluid domains is straightforward.

We notice that there exists one zero frequency mode in the $\phi-U$ formulation, i.e., constant velocity potential and zero displacements, denoted as $(\mathbf{0}^T\alpha\mathbf{i}^T\ \alpha)$ where $\mathbf{i}^T=(1\ 1\ \ldots\ 1)$, and the dimension of the vector $\mathbf{i}$ represents the number of the velocity potential unknowns left. Substituting the zero mode to Eq. (13) yields

$$\begin{aligned}\mathbf{K}_{ff}\mathbf{i}+\mathbf{K}_{af}^T &= \mathbf{0}\\ \mathbf{K}_{af}\mathbf{i}+\mathbf{K}_{aa} &= \mathbf{0}.\end{aligned} \qquad (A.2)$$

By assigning the mth eigensolution of Eq. (13) as $(\mathbf{U}_m^T\ (\boldsymbol{\Phi}_m^T+\alpha\mathbf{i}^T)\ \alpha)$ and using Eq. (A.2), we get the following equations:

$$\begin{aligned}-\omega_m^2\mathbf{M}_{ss}\mathbf{U}_m-(\mathbf{C}_{fs}^T\mathbf{i}+\mathbf{C}_{as}^T)\alpha\omega_m+\mathbf{K}_{ss}\mathbf{U}_m-\omega_m\mathbf{C}_{fs}^T\boldsymbol{\Phi}_m &= \mathbf{0}\\ -\omega_m^2\mathbf{M}_{ff}\boldsymbol{\Phi}_m-(\mathbf{M}_{ff}\mathbf{i}+\mathbf{M}_{af}^T)\alpha-\omega_m\mathbf{C}_{fs}\mathbf{U}_m+\mathbf{K}_{ff}\boldsymbol{\Phi}_m &= \mathbf{0}\\ (\mathbf{i}^T\mathbf{C}_{fs}+\mathbf{C}_{as})\mathbf{U}_m/\omega_m+(\mathbf{M}_{af}+\mathbf{i}^T\mathbf{M}_{ff})\boldsymbol{\Phi}_m+\alpha(\mathbf{i}^T\mathbf{M}_{ff}\mathbf{i}+2\mathbf{M}_{af}\mathbf{i}+\mathbf{M}_{aa}) &= \mathbf{0}.\end{aligned}$$

Replace α with $P/\rho_f\omega_m$ and compare with Eq. (A.1), we have

$$\begin{aligned}\mathbf{A}_p &= -(\mathbf{i}^T\mathbf{C}_{fs}+\mathbf{C}_{as})/\rho_f\\ \mathbf{B}_p &= (\mathbf{i}^T\mathbf{M}_{ff}+\mathbf{M}_{af})/\rho_f\\ \mathbf{D}_p &= -(\mathbf{i}^T\mathbf{M}_{ff}\mathbf{i}+2\mathbf{M}_{af}\mathbf{i}+\mathbf{M}_{aa})/\rho_f^2.\end{aligned}$$

Note that in Eqs. (7) and (9), we have $\sum_{i=1}^{N} h_i = 1$, $\sum_{i=1}^{N} a_i = 1$, $\sum_{i=1}^{N-1} a'_i = 1$, and $\sum_{i=1}^{N-1} h'_i = 1 - h_N$. Therefore it is obvious that we have $\mathbf{K}_{pp} = \mathbf{D}_p$, $\mathbf{K}_{ps} = \mathbf{A}_p$ and $\mathbf{C}_{pf} = \mathbf{B}_p$. By using the $P - \phi - U$ formulation, we in fact eliminate the zero frequency in the $\phi - U$ formulation, and both formulations have the same eigenvalues except at zero.

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